I J R B A T, Issue (XI) Vol (II) May 2023 : 378-380
A Double-Blind Peer Reviewed & Refereed Journal
e-ISSN 2347 – 517X
Original Article
INTERNATIONAL JOURNAL OF RESEARCHES IN BIOSCIENCES, AGRICULTURE AND TECHNOLOGY

INVERSION FORMULA OF GENERALIZED SIMPLIFIED FRACTIONAL FOURIER TRANSFORM

V. D. Sharma[1] and P. R. Langade[2]
[1]Dept. of Mathematics, Smt. Narsamma Arts, Commerce and Science College, Amravati, (MH), India.
[2]Dept. of Mathematics, Shri. Vasantrao Naik Mahavidyalaya, Dharni, Amravati, (MH), India.
Corresponding Email: vdsharma@hotmail.co.in, prlangade@gmail.com

Communicated : 26.01.2023 | Revision : 01.03.2023 & 10.03.2023 Accepted : 07.04.2023 | Published : 30.05.2023

ABSTRACT:
Inversion formula is used to retrieve original function to be transformed. Inversion formula of simplified fractional Fourier transform (SFRFT) says that for many types of functions it is possible to recover a function from its simplified fractional Fourier transform. If we know the function in frequency domain then we may reconstruct original function in time domain. Many differential equations can be solved by inversion formula. An application of inversion formula is Parseval's identity which states that sum or integral of product of functions is equal to the sum or integral of the product of its transform. The aim of this paper is to establish inversion formula and prove Parseval's identity for simplified fractional Fourier transform.

Keywords :- Simplified fractional Fourier transform, fractional Fourier transform, inversion formula, linear canonical transform, Parseval's identity.

INTRODUCTION :

The simplified fractional Fourier transform has been investigated in number of papers[1],[2] and has proved to be very useful in many application. It is the simplest form of the fractional Fourier transform (FRFT). Due to SFRFT, digital computation, optical implementation, graded-index (GRIN) medium implementation and radar system implementation become simplest than FRFT. Many application of FRFT such as removing chirp noise[3] and as fractional Hilbert transform[4] are based on the design of fractional filters. The fractional filter is the special case of fractional convolution[5]. We can replace the fractional Fourier transform with the Linear Canonical transform(LCT), generalize the fractional convolution into the canonical convolution and all the equal condition and all the LCT's with parameter $\{a,b,c,d\}$ will have same effects as the FRFT of order α for a design of a canonical filter if $a/_b = \cot\alpha$. For each type of implementation, what values of $\{a,b,c,d\}$ will satisfy $a/_b = \cot\alpha$ and have smallest implementation costs. Then LCT with this set of parameter can replace the FRFT with order α for this type of implementation, and it is the SFRFT. The purpose of this paper is to show that the so-called SFRFT is nothing more than a variation of the standard Fourier transform such as its inversion formula can be derived very easily from those of the Fourier transform by a single change of variable. To demonstrate this idea, we shall drive the inversion formula of the generalized SFRFT.

Preliminaries:-

We define the Fourier transform of a function f(t)

$$F(u) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} e^{-jut}\, f(t)dt$$

The inverse Fourier transform if

$$f(t) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} e^{jut}\, F(u)du$$

The simplified fractional Fourier transform with angle α of a signal f(t) is defined as

http://doi.org/10.29369/ijrbat.2023.02.1.0059

$[\text{SFRFT}(f(t))](u) = (j2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} exp\left(-jut + \frac{j}{2}t^2 cot\alpha\right) f(t)dt.$

Generalization of Simplified fractional Fourier transform:

The Generalization of simplified fractional Fourier transform with parameter α of $f(t)$ denoted by $O^{\alpha}_{F(1)}(f(t))$ performs a linear operation given by the integral transform,

$O^{\alpha}_{F(1)}(f(t)) = F_\alpha\{f(t)\}(u)$

$[\text{SFRFT } f(t)](u) = \int_{-\infty}^{\infty} f(t)K_\alpha(t,u)\, dt$

where, $K_\alpha(t,u) = (j2\pi)^{-\frac{1}{2}} e^{\left(-jut+\frac{j}{2}t^2 \cot(\alpha)\right)}$

The Test Function Space:

An infinitely differential complex valued smooth function on $\emptyset(R^n)$ belongs to $E(R^n)$, if for each compact $I \subset S_\alpha$, where, $S_\alpha = \{t\epsilon R^n, |t| \le a, a > 0\}, I\epsilon R^n$

$\gamma_{E,l}(\emptyset) = \sup_{t\epsilon I} |D_t^l \emptyset(t)| < \infty$, where $l = 1,2,3 \ldots$

Thus $E(R^n)$ will denote the space of all $\emptyset\epsilon E(R^n)$ with support contained in S_α. Note that the space E is complete and therefore a Frechlet's space. Moreover, we say that Generalization of simplified fractional Fourier transform if it is a member of E^*, the dual space of E.

Inversion formula of generalized simplified fractional Fourier transform

The generalized simplified fractional Fourier transform is given by

$[\text{SFRFT } f(t)](u) = \int_{-\infty}^{\infty} f(t)K_\alpha(t,u)\, dt$

where, $K_\alpha(t,u) = (j2\pi)^{-\frac{1}{2}} e^{\left(-jut+\frac{j}{2}t^2 \cot(\alpha)\right)}$

then its inverse f(t) is given by

$$f(t) = \int_{-\infty}^{\infty} F_\alpha(u)\overline{K_\alpha(t,u)}\, du$$

where $\overline{K_\alpha(t,u)} = (\frac{j}{2\pi})^{\frac{1}{2}} e^{jut-\frac{j}{2}t^2 cot\alpha}$

Solution:-

The generalized simplified fractional Fourier transform is given by

$[\text{SFRFT } f(t)](u) = \int_{-\infty}^{\infty} f(t)K_\alpha(t,u)\, dt$

where, $K_\alpha(t,u) = (j2\pi)^{-\frac{1}{2}} e^{\left(-jut+\frac{j}{2}t^2 \cot(\alpha)\right)}$

$$[\text{SFRFT}(f(t))](u) = (j2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} e^{-jut} e^{\frac{j}{2}t^2 cot\alpha} f(t)dt$$

put $e^{\frac{j}{2}t^2 cot\alpha} f(t) = g_\alpha(t)$

$$[\text{SFRFT}(f(t))](u) = (j2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} e^{-jut}\, g_\alpha(t)dt$$

$$[\text{SFRFT}(f(t))](u) = (j)^{-\frac{1}{2}} G_\alpha(u)$$

$$where\ G_\alpha(u) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} e^{-jut}\, g_\alpha(t)dt$$

This is the standard Fourier transform

Therefore the inverse Fourier transform

$$g_\alpha(t) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} e^{jut}\, G_\alpha(u)du$$

$$e^{\frac{j}{2}t^2 cot\alpha} f(t) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} e^{jut}\, j^{\frac{1}{2}}[\text{SFRFT}(f(t))](u)du$$

$$f(t) = (\frac{j}{2\pi})^{\frac{1}{2}} \int_{-\infty}^{\infty} e^{jut-\frac{j}{2}t^2 cot\alpha} F_\alpha(u)\, du$$

$$f(t) = \int_{-\infty}^{\infty} F_\alpha(u)\overline{K_\alpha(t,u)}\, du$$

where $\overline{K_\alpha(t,u)} = (\frac{j}{2\pi})^{\frac{1}{2}} e^{jut-\frac{j}{2}t^2 cot\alpha}$.

Parseval's identity for generalized simplified fractional Fourier transform

If $[\text{SFRFT}(f(t))](u) = F_\alpha(u)$ and $[\text{SFRFT}(g(t))](u) = G_\alpha(u)$ then

1) $\int_{-\infty}^{\infty} f(t)\overline{g(t)}\, dt = (\frac{-j}{2\pi})^{\frac{1}{2}} \int_{-\infty}^{\infty} F_\alpha(u)\overline{G_\alpha(u)}\, du$

2) $\int_{-\infty}^{\infty} |f(t)|^2\, dt = (\frac{-j}{2\pi})^{\frac{1}{2}} \int_{-\infty}^{\infty} |F_\alpha(u)|\, du$ Here bar indicate the conjugate function.

Proof:-

1)

$[\text{SFRFT}(f(t))](u) = F_\alpha(u) = (j2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} exp\left(-jut + \frac{j}{2}t^2 cot\alpha\right) f(t)dt$

$[\text{SFRFT}(g(t))](u) = G_\alpha(u)$

$$= (j2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} exp\left(-jut + \frac{j}{2}t^2 cot\alpha\right) g(t)dt$$

By the inversion formula of generalized SFRFT

$$g(t) = (\frac{j}{2\pi})^{\frac{1}{2}} \int_{-\infty}^{\infty} e^{jut-\frac{j}{2}t^2 cot\alpha} G_\alpha(u)\, du$$

$$\overline{g(t)} = (\frac{-j}{2\pi})^{\frac{1}{2}} \int_{-\infty}^{\infty} e^{jut-\frac{j}{2}t^2 cot\alpha} \overline{G_\alpha(u)}\, du$$

Now consider

$$\int_{-\infty}^{\infty} f(t)\overline{g(t)}\, dt$$

$$= (\frac{-j}{2\pi})^{\frac{1}{2}} \int_{-\infty}^{\infty} \overline{G_\alpha(u)} \int_{-\infty}^{\infty} f(t)\, e^{jut-\frac{j}{2}t^2 cot\alpha}\, dudt$$

http://doi.org/10.29369/ijrbat.2023.02.1.0059

$$\int_{-\infty}^{\infty} f(t)\overline{g(t)}\ dt = (\frac{-j}{2\pi})^{\frac{1}{2}} \int_{-\infty}^{\infty} \overline{G_\alpha(u)}\, F_\alpha(u)\, du$$

2) putting $f(t) = g(t), \quad F_\alpha(u) = G_\alpha(u)\ and\ \overline{F_\alpha(u)} = \overline{G_\alpha(u)}$

by using the above result

$$\int_{-\infty}^{\infty} f(t)\overline{g(t)}\ dt = (\frac{-j}{2\pi})^{\frac{1}{2}} \int_{-\infty}^{\infty} \overline{G_\alpha(u)}\, F_\alpha(u)\, du$$

$$\int_{-\infty}^{\infty} f(t)\overline{f(t)}\ dt = (\frac{-j}{2\pi})^{\frac{1}{2}} \int_{-\infty}^{\infty} \overline{F(u)}\, F_\alpha(u)\, du$$

$$\int_{-\infty}^{\infty} |f(t)|^2\ dt = (\frac{-j}{2\pi})^{\frac{1}{2}} \int_{-\infty}^{\infty} |F_\alpha(u)|\, du.$$

CONCLUSION :

In this paper, we have proved the inversion formula of generalized simplified fractional Fourier transform and also proved the Parseval's identity of generalized simplified fractional Fourier transform

REFERENCES:

Pei, S.C. and Ding, J.J., Simplified fractional Fourier Transform, Journal of the optical society of America A. (2000).

Sharma, V. D., Introduction to Generalized Simplified Fractional Fourier Transform, IJSR, 2313-7064, (2012).

Ozaktas,H. M., Z.,Kutay,M. A., Arikan, O. and Onural, L., Optimal filter in fractional Fourier domains, IEEE Trans. Signal processing, 1129-1143(1997).

Lohmann, A. W., Mendlovic, D. and Zalevsky, Z., Fractional Hilbert transform, Opt. Lett., 281-283(1996).

Ozaktas,H. M., Barshan, B., Mendlovic, D. and Onural, L., Convolution, filtering, and multiplexing in fractional Fourier domains and their rotation to chirp and wavelet transform, J. Opt. Soc. Am., 547-559 (1994).

Kumar Sanjay, Convolution, Product and Correlation Theorems for Simplified Fractional Fourier Transform: A Mathematical Investigation, Research Journal of Quantum Computations and Physical Systems Vol 1(1), (2018).

Bhosale, B.N., Integral transformation of Generalized functions. Discovery publishing hour, (2005).

Zemanian, A. H., Generalized Integral Transform, Interscience Publication, New York, (1968).

Sharma, V.D.,Introduction to Fourier-Laplace Transform, International Journal of Management and Applied Science, Vol.2, Issue-6(2016).

http://doi.org/10.29369/ijrbat.2023.02.1.0059